Mechanics: Gravitation

FIZIKA SPhO Training

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Contents

1 Notes

The gravitational force is one of the four fundamental forces in nature. It is a mutual interaction between all massive particles, and it is always attractive in nature.

1.1 The Basic Quantities

In gravitation, the four basic quantities you will deal with are:

1. Gravitational Force, $\mathbf{F}_g$
2. Gravitational Field, $\mathbf{g}$
3. Gravitational Potential Energy, U_g
4. Gravitational Potential, ϕ_g

Note that the first two are vectors while the other two are scalars.

1.1.1 Gravitational Force and Gravitational Field

Consider two point masses M_1 and M_2 located at position vectors $\mathbf{r}_1$ and $\mathbf{r}_2$. Let the separation vector be $\mathbf{r} = \mathbf{r}_2 - \mathbf{r}_1$.

The gravitational forces by each point mass on the other mass are given by

$$\mathbf{F}_{g,1\,by\,2} = \frac{GM_1M_2}{|\mathbf{r}|^3}\mathbf{r} \tag{1}$$

$$\mathbf{F}_{g,2\,by\,1} = -\frac{GM_1M_2}{|\mathbf{r}|^3}\mathbf{r} \tag{2}$$

The gravitational field can be thought of as the gravitational force per unit mass. Let the mass M be placed at the origin. The gravitational field at a position vector $\mathbf{r}$ away is given by

$$\mathbf{g} = -\frac{GM}{|\mathbf{r}|^3}\mathbf{r} \tag{3}$$

1.1.2 Gravitational Potential Energy and Gravitational Potential

Consider two point masses again. The gravitational potential energy is given by

$$U_g = -\frac{GM_1M_2}{|\mathbf{r}|} \tag{4}$$

The gravitational potential can be thought of as the gravitational potential energy per unit mass. Thus, the gravitational potential at a position vector $\mathbf{r}$ away is given by

$$\phi_g = -\frac{GM}{|\mathbf{r}|} \tag{5}$$

The **negative signs** in Equations (4) and (5) are very important! The signs are negative as the gravitational force is always an attractive interaction.

1.1.3 Relationships Between Quantities

As we have studied before, force is the negative of the derivative of potential energy. Thus,

$$F_g = -\frac{dU_g}{dr} \tag{6}$$

Conversely, to get gravitational potential energy from gravitational force,

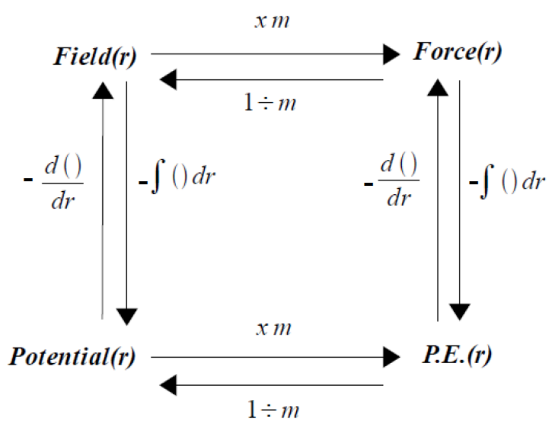
$$U_g = -\int_{\infty}^{\mathbf{r}} \mathbf{F}_g \cdot d\mathbf{r} \tag{7}$$

Since the gravitational field and gravitational potential can be thought of as the gravitational force and gravitational potential energy per unit mass respectively, we also have

$$g = -\frac{d\phi_g}{dr} \tag{8}$$

$$\phi_g = -\int_{\infty}^{\mathbf{r}} \mathbf{g} \cdot d\mathbf{r} \tag{9}$$

A helpful image for remembering this is the square below:



1.2 Circular Orbits

You should already be familiar with this from H2 physics - this is just a recap.

1.2.1 Period, Speed and Radius of Circular Orbits

Consider a satellite of mass m in a **circular orbit** around a planet of mass M. We shall assume $M >> m$ (which is justifiable for a normal satellite and normal planet), so that the planet is approximately stationary at the centre of the orbit.

Since the gravitational force provides for the centripetal force,

$$F_c = F_g \quad \Rightarrow \quad \frac{mv^2}{R} = \frac{GMm}{R^2} \quad \Rightarrow \quad v = \sqrt{\frac{GM}{R}} \tag{10}$$

Thus, from circular motion,

$$\omega = \frac{v}{R} = \sqrt{\frac{GM}{R^3}}, \; T = \frac{2\pi}{\omega} = 2\pi\sqrt{\frac{R^3}{GM}} \tag{11}$$

Remember that none of these quantities depend on the mass of the satellite, m!

We are mostly interested in **geostationary orbits.** This is a special type of orbit centered around Earth, where the satellite remains **fixed** above a specific point on the Earth's equator.

The period of the geostationary orbit is equal to the period of the Earth's rotation:

$$T_{geostationary} = T_{Earth} \approx 24\,\text{h} \tag{12}$$

You can show that the radius of geostationary orbit is around $R_{geostationary} = 42000\,\text{km}$.

1.2.2 Energy of Circular Orbits

The total energy of a circular orbit consists of kinetic energy and gravitational potential energy. As there are no external forces acting on the satellite-planet system, **total energy is conserved.**

We may calculate the different forms of energy as such:

$$K = \frac{1}{2}mv^2 = \frac{GMm}{2R},\ U_g = -\frac{GMm}{R},\ E = K + U_g = -\frac{GMm}{2R} = \text{constant} \tag{13}$$

Note that the total energy $E < 0$. This implies that the orbit is **bounded.** We'll see more on what bounded orbits mean in Section 1.6.

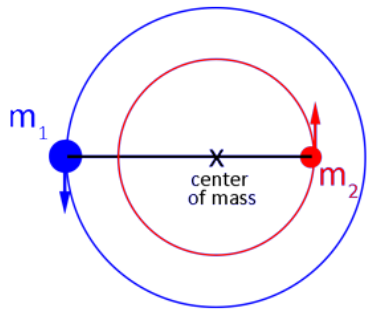
Only for circular orbits, we may write:

$$K = -E,\ U_g = 2E,\ U_g = -2K \tag{14}$$

These relations may be useful if you need to find one form of energy given another.

1.2.3 Binary Star Systems

If the masses of the two gravitationally interacting objects are comparable (i.e. M is not $>> m$), then both objects will revolve around each other, **about their CM.** The orbits are concentric circles, as shown in the diagram below.



Firstly, note that the angular frequency of both orbits must be the same, so that the CM stays at the same position. (The CM cannot move as there are no net external forces on the system.)

$$\omega_1 = \omega_2 = \omega \tag{15}$$

Let the radius of orbit of m_1 be r_1, and that of m_2 be r_2, and $r = r_1 + r_2$. Then,

$$m_1 r_1 \omega_1^2 = \frac{Gm_1 m_2}{r^2},\ m_2 r_2 \omega_2^2 = \frac{Gm_1 m_2}{r^2} \tag{16}$$

From the CM relation, we have

$$\frac{r_2}{r_1} = \frac{m_1}{m_2} \tag{17}$$

Using Equations (15), (16) and (17), we can obtain that

$$\omega = \sqrt{\frac{G(m_1 + m_2)}{r^3}} \tag{18}$$

We can analyse this using **reduced mass** as well. Recall that the system is equivalent to a mass $\mu = \frac{m_1 m_2}{m_1 + m_2}$ interacting with a mass $M = m_1 + m_2$ at rest. You can then recover the same results. (Try it yourself as an exercise!)

1.3 Elliptical Orbits

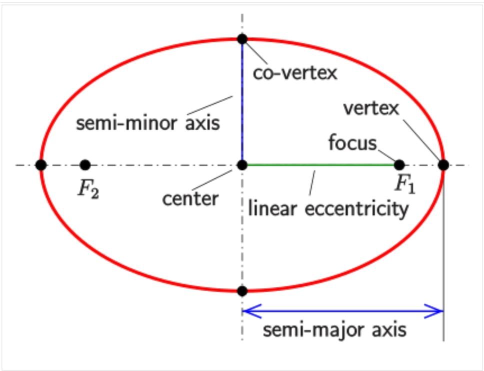
In reality, most orbits are not circular. They are instead **elliptical!**

1.3.1 Ellipse Geometry

An ellipse centred at the origin can be described by the Cartesian equation

$$\frac{x^2}{a^2} + \frac{y^2}{b^2} = 1 \tag{19}$$

where a and b ($a > b$) are the semi-major and semi-minor axes of the ellipse respectively.



The **foci** (labelled F_1 and F_2) of the ellipse are located at $(\pm c, 0)$, where

$$c = \sqrt{a^2 - b^2} \tag{20}$$

The value of c is also known as the **linear eccentricity.**

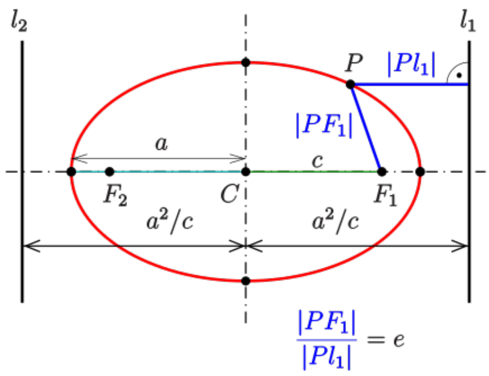
The foci are special points such that for all points P on the ellipse, they satisfy

$$PF_1 + PF_2 = 2a = \text{constant} \tag{21}$$

The **eccentricity** ε (different from linear eccentricity!) of the ellipse is defined as

$$\varepsilon = \frac{c}{a} = \sqrt{1 - \frac{b^2}{a^2}} \tag{22}$$

The **directrix** of the ellipse is a special line lying *outside* the ellipse.



As per the diagram above, the directrix is located at

$$x_{directrix} = \frac{a^2}{c} \tag{23}$$

and it satisfies

$$\frac{PF_1}{Pl_1} = \varepsilon \tag{24}$$

for all points P on the ellipse.

You can visualise for yourself that that a circle has $\varepsilon \to 0^+$ and a line has $\varepsilon \to 1^-$.

1.3.2 Polar Equation of Elliptical Orbits

Consider a satellite of mass m in an **elliptical orbit** around a planet of mass M. Again, we assume that $M >> m$ so that the planet is approximately stationary.

The polar equation of the ellipse, with the origin at the position of the planet, is

$$r = \frac{p}{1 - \varepsilon \cos\theta}, \ 0 < \varepsilon < 1, \ p > 0 \tag{25}$$

We may transform Equation (25) into Cartesian coordinates. Recall that $r^2 = x^2 + y^2$, and $x = r\cos\theta$, $y = r\sin\theta$. Hence,

$$r = p + \varepsilon r\cos\theta \quad \Rightarrow \quad r^2 = x^2 + y^2 = (p + \varepsilon x)^2 \quad \Rightarrow \quad \frac{\left(x - \frac{p\varepsilon}{1-\varepsilon^2}\right)^2}{\left(\frac{p}{1-\varepsilon^2}\right)^2} + \frac{y^2}{\left(\frac{p^2}{1-\varepsilon^2}\right)} = 1 \tag{26}$$

Recall from H2 math that this is an ellipse with origin shifted to the right by $\frac{p\varepsilon}{1-\varepsilon^2}$ units along the positive x-axis. Comparing to the form of the ellipse equation in Equation (19), we have

$$a = \frac{p}{1-\varepsilon^2}, \ b = \frac{p}{\sqrt{1-\varepsilon^2}} \tag{27}$$

Thus, we may define this constant p from Equation (25) as

$$p = \frac{b^2}{a} = \frac{a^2 - c^2}{a} \tag{28}$$

1.3.3 Apsides of Elliptical Orbits

Based on Equation (25), we can see that by varying θ,

$$r_{max} = \frac{p}{1-\varepsilon},\ r_{min} = \frac{p}{1+\varepsilon} \tag{29}$$

The *existence* of a maximum and minimum radial distance means that the orbit is **bounded.**

The maximum and minimum distances occur at the points named the **apsides.** There are two types of apsides - the **periapsis** (nearest point) and the **apoapsis** (furthest point).

Remark. In some books, you may see the words "perihelion" and "aphelion", or "perigee" and "apogee". These are equivalent to the periapsis and apoapsis for the specific cases of a body orbiting the Sun and the Earth repsectively.

1.3.4 Angular Momentum of Elliptical Orbits

Recall from the rotation handout that the angular momentum is defined as

$$\mathbf{L} = m\mathbf{r} \times \mathbf{v} \tag{30}$$

From Equations (1) and (2), the gravitational force acts purely in the radial direction (with the planet/Sun that is being revolved around as the origin). Thus, it contributes to *no net torque* around the central axis of the planet/Sun. Hence, **angular momentum is conserved!**

For an elliptical orbit, we are most interested in the angular momentum at the apsides, since the cross product in Equation (30) is easiest to calculate as $\mathbf{r} \perp \mathbf{v}$.

Thus, by COAM, letting subscripts p and a denote the periapsis and apoapsis respectively,

$$mr_p v_p = mr_a v_a \quad \Rightarrow \quad r_p v_p = r_a v_a \tag{31}$$

1.3.5 Energy of Elliptical Orbits

Similar to a circular orbit, the total energy of an elliptical orbit consists of kinetic energy and gravitational potential energy. Again, as there are no external forces acting on the system, total energy is conserved. However, the **radial distance isn't constant!**

The total energy for an elliptical orbit of semi-major axis a is instead given by

$$E = K + U_g = -\frac{GMm}{2a} = \text{constant} \tag{32}$$

The proof of Equation (32) is in the Appendix. However, it suffices to just quote it.

In many elliptical orbit questions, you will need to use Equation (31) and (32) in tandem. The example below illustrates how.

Example 1.1 (Ricardo)**.** A satellite of mass m_s is in an elliptical orbit around a planet of mass m_p ($m_p >> m_s$), which is located at one focus of the ellipse. The satellite has a speed v_a at the distance r_a when it is furthest from the planet. The distance of closest approach is r_p. Find the ratio $\frac{r_a}{r_p}$ in terms of G, m_p, v_a and r_a.

Applying COAM at the periapsis and apoapsis,

$$r_p v_p = r_a v_a \quad \Rightarrow \quad v_p = \frac{r_a v_a}{r_p}$$

Applying COE at the periapsis and apoapsis,

$$\frac{1}{2}mv_p^2 - \frac{Gm_pm}{r_p} = \frac{1}{2}mv_a^2 - \frac{Gm_pm}{r_a} \quad \Rightarrow \quad v_p^2 - v_a^2 = 2Gm_p\left(\frac{1}{r_p} - \frac{1}{r_a}\right)$$

To eliminate v_p, substitute the first equation into the second:

$$\left(\frac{r_a^2}{r_p^2} - 1\right)v_a^2 = 2Gm_p\left(\frac{r_a - r_p}{r_pr_a}\right) \quad \Rightarrow \quad \left(\frac{r_a^2 - r_p^2}{r_p^2}\right)v_a^2 = 2Gm_p\left(\frac{r_a - r_p}{r_pr_a}\right)$$

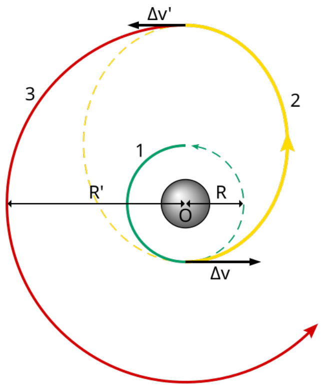
$$\Rightarrow \quad \left(\frac{(r_a - r_p)(r_a + r_p)}{r_p^2}\right)v_a^2 = 2Gm_p\left(\frac{r_a - r_p}{r_pr_a}\right) \quad \Rightarrow \quad \left(\frac{r_a + r_p}{r_p}\right)v_a^2 = \frac{2Gm_p}{r_a}$$

Let our desired ratio $\frac{r_a}{r_p} = x$. Thus,

$$(x+1)\,v_a^2 = \frac{2Gm_p}{r_a} \quad \Rightarrow \quad x = \frac{2Gm_p}{r_av_a^2} - 1$$

1.3.6 Hohmann Transfer Orbit

The Hohmann transfer Orbit is a type of elliptical orbit, used as an intermediate to transfer a satellite from one circular orbit to another.



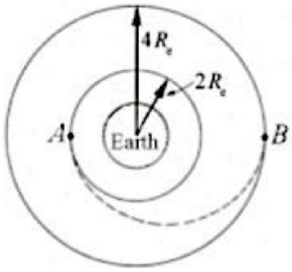
The Hohmann transfer orbit is labelled "2" in the diagram above. (It actually only consists of *half* the elliptical orbit.)

The key features of the Hohmann transfer orbit are:

1. It consists of **two speed changes,** $\Delta v > 0$ and $\Delta v' < 0$, at the start and end respectively. These are usually achieved by rocket propulsion.
2. The semi-major axis of the Hohmann transfer orbit is given by $a = \frac{R+R'}{2}$.
3. The satellite starts at the periapsis and ends at the apoapsis.

Remark. In SPhO, you are **expected** to know what this refers to. That is, you might *not* be given a diagram for Hohmann transfer orbit questions!

Example 1.2 (Ricardo)**.** A space vehicle is in a circular orbit around the Earth of radius R_e and mass M_e. The mass of the vehicle is $m_s << M_e$ and the radius of the orbit is $2R_e$. We want to transfer the vehicle to a circular orbit of radius $4R_e$. (i) Find the minimum energy expenditure for the Hohmann transfer. (ii) What changes in speed are required at points A and B in the Hohmann transfer (refer to the diagram below)?



(i) We may just consider the *change* in total energy of the two orbits:

$$\Delta E_{min} = E_f - E_i = -\frac{GM_e m_s}{2\,(4R_e)} - \left(-\frac{GM_e m_s}{2\,(2R_e)}\right) = \frac{GM_e m_s}{8R_e}$$

The positive sign makes sense as we need to *supply* energy for the transfer.

(ii) The semi-major axis of the Hohmann transfer orbit is

$$a = \frac{2R_e + 4R_e}{2} = 3R_e$$

The total energy of an elliptical orbit with this semi-major axis is

$$E_{new} = -\frac{GM_e m_s}{2a} = -\frac{GM_e m_s}{2\,(3R_e)} = -\frac{GM_e m_s}{6R_e}$$

The total energy is conserved, so we may calculate the KE at points A and B using the total energy and the GPE.

At the periapsis (point A),

$$-\frac{GM_e m_s}{6R_e} = -\frac{GM_e m_s}{2R_e} + \frac{1}{2} m_s v_A^2 \quad \Rightarrow \quad v_A = \sqrt{\frac{2GM_e}{3R_e}}$$

and at the apoapsis (point B),

$$-\frac{GM_e m_s}{6R_e} = -\frac{GM_e m_s}{4R_e} + \frac{1}{2} m_s v_B^2 \quad \Rightarrow \quad v_B = \sqrt{\frac{GM_e}{6R_e}}$$

The speeds in the circular orbits are $v_{A,circular} = \sqrt{\frac{GM_e}{2R_e}}$ and $v_{B,circular} = \sqrt{\frac{GM_e}{4R_e}}$. Thus, the respective changes in speed are

$$\Delta v_A = \sqrt{\frac{2GM_e}{3R_e}} - \sqrt{\frac{GM_e}{2R_e}} = \left(\sqrt{\frac{2}{3}} - \sqrt{\frac{1}{2}}\right)\sqrt{\frac{GM_e}{R_e}}$$

$$\Delta v_B = \sqrt{\frac{GM_e}{6R_e}} - \sqrt{\frac{GM_e}{4R_e}} = \left(\sqrt{\frac{1}{6}} - \sqrt{\frac{1}{4}}\right)\sqrt{\frac{GM_e}{R_e}}$$

As expected, $\Delta v_A > 0$ and $\Delta v_B < 0$.

1.4 Kepler's Laws

Kepler's three laws are essential to our understanding of gravitation and planetary motion. The detailed proofs of each law can be found in the Appendix.

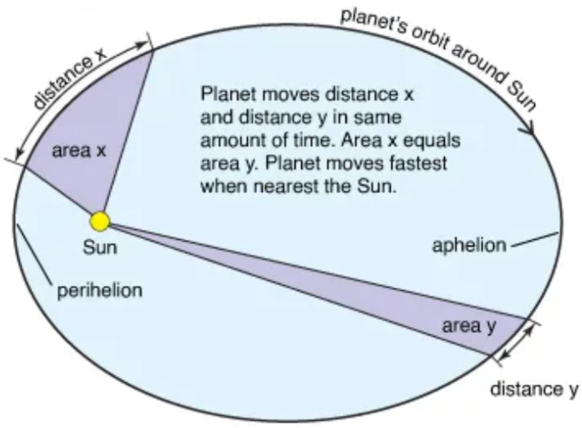
1.4.1 Kepler's First Law

Statement: The orbit of every planet is an **ellipse** with the Sun at one of the two foci.

This is self-explanatory. The other focus of the ellipse is what we call the **empty focus.**

1.4.2 Kepler's Second Law

Statement: A line joining the planet to the Sun sweeps equal areas in equal times.



This means that $\frac{dA}{dt}$ = constant, or alternatively, by integrating over one period, we may write

$$\pi ab = \frac{LT}{2m} \tag{33}$$

where L is the angular momentum about the axis of the Sun, T is the period, and m is the mass of the planet.

This is useful for problems where the area swept out by an orbit can be easily calculated.

1.4.3 Kepler's Third Law

Statement: The square of the orbital period is proportional to the cube of the semi-major axis.

Mathematically, we may write

$$T = \frac{2\pi a^{\frac{3}{2}}}{\sqrt{GM}} \tag{34}$$

where T is the period, a is the semi-major axis, and M is the mass of the Sun.

Clearly, Kepler's Third Law only applies for **bounded orbits,** as we can only define a period for a bounded orbit and not an orbit that goes to infinity.

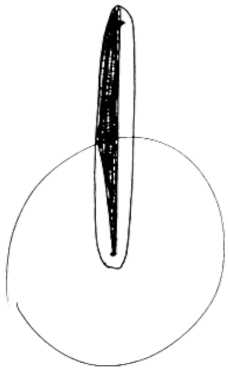
You'll usually need to apply multiple laws in a problem. Let's look at an example.

Example 1.3 (Kevin Zhou)**.** An object of mass m is dropped from rest at a distance R from above the Earth's surface, where R is also the radius of the Earth. Given that the mass of the Earth is $M >> m$, how long does it take to hit the Earth's surface?

The solution to this uses an ingenious trick in order to utilise the idea of elliptical orbits!

Notice that the time will not change much if the mass was given a small kick in the horizontal direction before release. The orbit will hence become a very thin ellipse, with $a \approx R$, with one focus very near the centre of the Earth, by Kepler's First Law.

A diagram of the elliptical orbit is shown below:



The area swept out by the portion of the orbit we are interested in is the shaded area.

Let the full orbit have period T. Recall that the area of an ellipse is πab where a and b are the semi-major and semi-minor axes respectively. Then, by Kepler's Second Law, using the ratio of areas, the time taken is

$$t = T\left(\frac{\frac{\pi ab}{4} + \frac{ab}{2}}{\pi ab}\right) = T\left(\frac{1}{4} + \frac{1}{2\pi}\right)$$

by summing up a quarter of an ellipse and a right triangle.

Now, the time T is given by Kepler's Third Law:

$$T = \frac{2\pi a^{\frac{3}{2}}}{\sqrt{GM}} = \frac{2\pi R^{\frac{3}{2}}}{\sqrt{GM}}$$

Hence, we obtain the final answer,

$$t = \left(\frac{\pi}{2} + 1\right)\sqrt{\frac{R^3}{GM}}$$

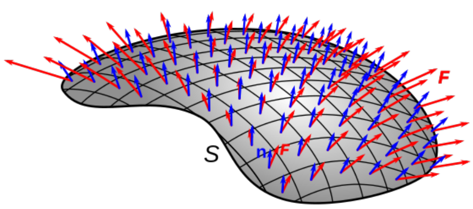
You should understand how all three of Kepler's Laws came into play in this example.

1.5 Ideas

Finding gravitational fields can be tricky. We shall introduce three ideas that may help.

1.5.1 Gravitational Flux and Gauss' Law

Loosely speaking, flux is defined as the "number of field lines passing through an area".



Mathematically speaking, the flux of a vector field $\mathbf{F}$ is defined using a **surface integral:**

$$\Phi_F = \iint_A \mathbf{F} \cdot d\mathbf{A} \tag{35}$$

Here, note that the subscript A represents that the integral is carried out over an area. Also, $d\mathbf{A}$ is an infinitesimal area **vector** - refer to the blue arrows in the diagram above!

When we calculate **gravitational flux,** we usually choose a surface that is **closed**, meaning that it encloses a finite, non-zero volume. Additionally, in this case, our vector field $\mathbf{F}$ is just the gravitational field, $\mathbf{g}$. We hence use the following notation for gravitational flux:

$$\Phi_g = \oiint_A \mathbf{g} \cdot d\mathbf{A} \tag{36}$$

Gauss' Law for Gravity relates Equation (36) to the **mass enclosed** by the closed surface:

$$\oiint_A \mathbf{g} \cdot d\mathbf{A} = -4\pi G m_{enclosed} \tag{37}$$

Don't be intimidated by the surface integrals! In Physics Olympiad, you'll only be applying Equation (37) to situations with **high symmetry.** These are situations where $|\mathbf{g}|$ is constant over your chosen surface, and $\mathbf{g}$ is purely parallel/anti-parallel to $d\mathbf{A}$, allowing you to simplify the dot product $\mathbf{g} \cdot d\mathbf{A}$.

Remark. Be careful of the sign conventions! For the area vector, outward is defined as positive while inward is defined as negative. Thus, outward flux is defined as positive while inward flux is defined as negative!

These surfaces with high symmetry are known as **Gaussian surfaces.** You'll need to spot **three** types of Gaussian surfaces in Olympiad problems, corresponding to different symmetries:

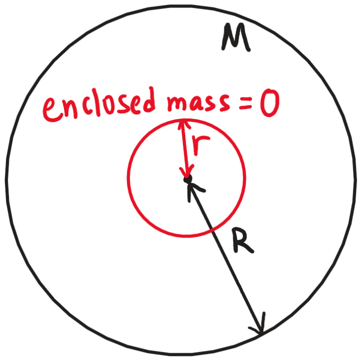
1. **Sphere:** Radially symmetric gravitational fields
2. **Cylinder:** Cylindrically symmetric gravitational fields
3. **Pillbox:** Planarly symmetric gravitational fields

Example 1.4. Determine the magnitude of the gravitational fields due to the following mass distributions. (i) A hollow sphere of mass M and radius R, at a point a distance r from the centre of the sphere. (ii) A solid, uniform sphere of mass M and radius R, at a point a distance r from the centre of the sphere. (iii) An infinitely long rod with uniform linear mass density μ at a point a distance x from the rod. (iv) An infinite plane with uniform area mass density σ at a point a distance x from the surface.

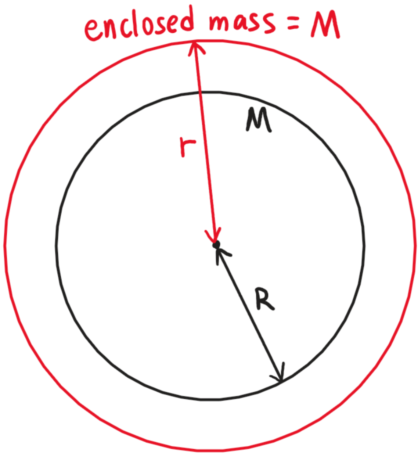
While you could have done all four parts just using naive integration (try it yourself as an exercise!), this example shows you how to apply each type of Gaussian surface to find the gravitational fields in a more elegant manner.

(i) Firstly, note that the behaviour will be different for $r < R$ and $r \geq R$, as the relationship between $m_{enclosed}$ and r differs in both cases. (Don't forget to consider both cases!) We choose our Gaussian surface to be a **concentric sphere of radius** r**,** due to the radial symmetry.

For the former, the enclosed mass is always 0, as the Gaussian surface lies inside the hollow sphere, which has no mass inside. Thus, by Gauss' Law, $g = 0$.



For the latter, the enclosed mass is always M, as the Gaussian surface completely encloses the whole sphere of mass M.



Due to symmetry, $\mathbf{g}$ always points radially inwards, and is anti-parallel with $d\mathbf{A}$ (pointing radially outwards from the Gaussian sphere). Hence, $\mathbf{g} \cdot d\mathbf{A} = -g\,dA$. Thus, by Gauss' Law,

$$\oiint_A \mathbf{g} \cdot d\mathbf{A} = -g\left(4\pi r^2\right) = -4\pi G m_{enclosed} = -4\pi GM \quad \Rightarrow \quad g = \frac{GM}{r^2}$$

Hence, the final answer for the gravitational field is

$$g(r) = \begin{cases} 0, \;\; r < R \\ \frac{GM}{r^2}, \;\; r \geq R \end{cases}$$

(ii) Again, note that the behaviour will be different for $r < R$ and $r \geq R$. Again, we choose our Gaussian surface to be a **concentric sphere of radius r.**

For the former, since the density is constant, the enclosed mass scales by r^3, and is $m_{enclosed} = \frac{r^3}{R^3}M$. Thus, by Gauss' Law,

$$\oiint_A \mathbf{g} \cdot d\mathbf{A} = -g\left(4\pi r^2\right) = -4\pi G m_{enclosed} = -4\pi GM\frac{r^3}{R^3} \quad \Rightarrow \quad g = \frac{GMr}{R^3}$$

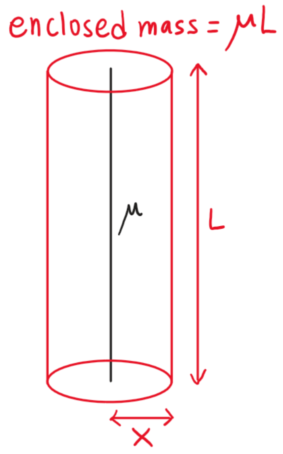
For the latter, the enclosed mass is just M. Thus, by Gauss' Law,

$$\oiint_A \mathbf{g} \cdot d\mathbf{A} = -g\left(4\pi r^2\right) = -4\pi G m_{enclosed} = -4\pi GM \quad \Rightarrow \quad g = \frac{GM}{r^2}$$

Hence, the final answer for the gravitational field is

$$g(r) = \begin{cases} \frac{GMr}{R^3}, & r < R \\ \frac{GM}{r^2}, & r \geq R \end{cases}$$

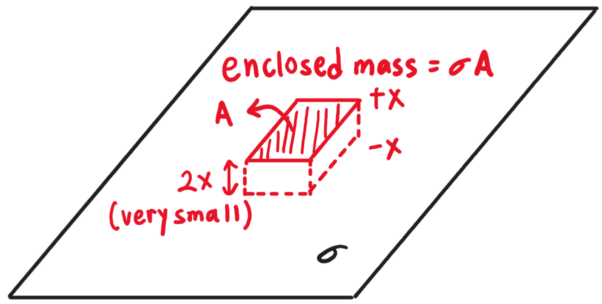
(iii) The infinite nature of the rod means that there should not be a component of the gravitational field parallel to the rod's axis. (You can think of there being an equivalent mass on the other side to cancel out any parallel component, since it is infinite.) We choose our Gaussian surface to be a **coaxial cylinder of length L and radius x,** due to the cylindrical symmetry.



Due to symmetry, $\mathbf{g}$ always points radially inwards towards the axis of the cylinders, and is anti-parallel with $d\mathbf{A}$ (pointing radially outwards from the curved surface area of the Gaussian cylinder). Hence, $\mathbf{g} \cdot d\mathbf{A} = -g\, dA$. Thus, by Gauss' Law,

$$\oiint_A \mathbf{g} \cdot d\mathbf{A} = -g\,(2\pi x L) = -4\pi G m_{enclosed} = -4\pi G \mu L \quad \Rightarrow \quad g(x) = \frac{2G\mu}{x}$$

(iv) Similarly, the infinite nature of the plane means that there should not be a component of the gravitational field parallel to the plane. We choose our Gaussian surface to be a **"pillbox" (a very short box) of cross-sectional area A, with its flat faces at $\pm x$ parallel to the plane,** due to the planar symmetry.



Due to symmetry, $\mathbf{g}$ always points perpendicularly inwards towards the plane, and is anti-parallel with the $d\mathbf{A}$ on the larger surfaces, and perpendicular to the $d\mathbf{A}$ on the smaller surfaces. That is, the smaller surfaces contribute no gravitational flux, and for the larger surfaces, $\mathbf{g} \cdot d\mathbf{A} = -g\, dA$. Thus, by Gauss' Law,

$$\oiint_A \mathbf{g} \cdot d\mathbf{A} = -g\,(2A) = -4\pi G m_{enclosed} = -4\pi G \sigma A \quad \Rightarrow \quad g(x) = 2\pi G \sigma$$

Surprisingly, this set-up produces a uniform gravitational field!

1.5.2 Newton's Shell Theorems

In Example 1.4 (i) and (ii), we have proven the following, called **Newton's Shell Theorems:**

1. **Inside** a uniform spherical shell, there is **no gravitational field** everywhere.
2. **Outside** a uniform spherical shell of total mass m, the gravitational field is the same as if the shell was replaced by a **point mass** m **at its centre.**

These statements are useful to simplify the calculations of gravitational fields in shells.

1.5.3 Superposition and Negative Mass

Sometimes, certain objects may have "holes" in them, whereby mass has been removed. Calculating the gravitational fields may be difficult, since symmetry is broken.

However, the good news is that gravitational fields obey the **principle of superposition!** This means that fields add linearly as vectors. Thus, you may calculate the gravitational fields of each piece of mass separately (using Gauss' Law or naive integration), and add them together!

The concept of **negative mass** can be applied to holes. Using the principle of superposition, we may find the gravitational field as if the hole was filled normally, and then add up the field produced by a negative mass filling up the hole.

Example 1.5. Consider the infinite plane in Example 1.4 (iv), except a hole of radius R is now present, centered at the origin. Find the gravitational field along the axis of the hole.

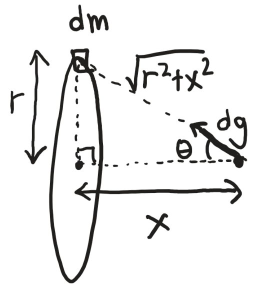
Using the principle of superposition,

$$\mathbf{g} = \mathbf{g}_{plane} + \mathbf{g}_{hole}$$

where $\mathbf{g}_{hole}$ is produced by a hole of the same dimensions, but a negative mass density $-\sigma$.

We already found $\mathbf{g}_{plane}$ in Example 1.4 (iv). To find $\mathbf{g}_{hole}$, let us perform naive integration, by treating the circular hole as many rings together.

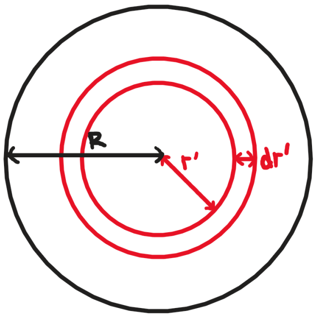
Consider a ring of radius r and mass m. We want to find the gravitational field along its axis, at a distance x away from its centre.



Clearly, after integrating across the ring, only the component $dg\cos\theta$ will survive. From the diagram, $\cos\theta = \frac{x}{\sqrt{r^2+x^2}}$. Thus,

$$g_{ring} = \int dg\,\cos\theta = \int \frac{G\,dm}{r^2+x^2}\frac{x}{\sqrt{r^2+x^2}} = \frac{Gx}{(r^2+x^2)^{\frac{3}{2}}}\int dm = \frac{Gmx}{(r^2+x^2)^{\frac{3}{2}}}$$

Now, we integrate across all the rings that form the circular hole.



The mass of each small ring is $dm = -\sigma\, dA = -2\pi r'\sigma\, dr'$, accounting for negative mass. Thus,

$$g_{hole} = \int \frac{Gx\, dm}{(r'^2 + x^2)^{\frac{3}{2}}} = -\int_0^R \frac{2\pi G\sigma x r'\, dr'}{(r'^2 + x^2)^{\frac{3}{2}}} = -\frac{2\pi G\sigma x}{\sqrt{x^2 + R^2}}$$

Hence, by the principle of superposition,

$$g(x) = 2\pi G\sigma - \frac{2\pi G\sigma x}{\sqrt{x^2 + R^2}} = 2\pi G\sigma \left(1 - \frac{x}{\sqrt{x^2 + R^2}}\right)$$

Of course, you could have solved this without negative mass, by just integrating the gravitational field of the rings from R to ∞. However, this method is much more flexible, as it can apply to other situations where naive integration is difficult due to loss of symmetry!

1.6 References

An Introduction to Mechanics by *Kleppner & Kolenkow*

Gravitation Handout by *Kevin Zhou*

Competitive Physics by *Wang & Ricardo*

University Physics by *Young & Freedman*

An Introduction to Classical Mechanics by *David Morin*

2 Problems

Problems are arranged in roughly increasing difficulty. Take $G = 6.67 \times 10^{11}\,\mathrm{Nm^2/kg^2}$ if needed.

Problem 2.1 (SPhO 2009)**.** An object is fired vertically upwards from the surface of the Earth (of radius R_E) with an initial speed v_i that is comparable to but less than the escape speed v_{esc}. (i) Show that the object attains a maximum height h given by

$$h = \frac{R_E v_i^2}{v_{esc}^2 - v_i^2}$$

(ii) A space vehicle is launched vertically upwards from the Earth's surface with an initial speed of 8.76 km/s, which is less than the escape speed of 11.2 km/s. Find the maximum height that it attains. (iii) A meteorite falls towards the Earth. It is essentially at rest with respect to the Earth when it is at a height of 2.51×10^7 m. Determine the speed at which the meteorite strikes the Earth. (iv) Assume that a baseball is tossed up with an initial speed that is **very small** compared to the escape speed. Show that the maximum height attained by the baseball is

$$h = \frac{v_i^2}{2g}$$

where g is the acceleration due to free fall on the surface of the Earth. Show also that the equation from part (i) is consistent with this equation.

Problem 2.2 (SPhO 2008)**.** A satellite of mass 1000 kg is in an elliptical orbit around the Earth. The minimum height of the orbit above the Earth's surface is 250 km (at perigee) and the maximum height above the Earth's surface is 1300 km (at apogee). (i) Qualitatively sketch the variation of the kinetic energy of the satellite with the height above the Earth's surface from perigee to apogee. (ii) If the velocities of the satellite at apogee and at perigee are v_a and v_p respectively, using conservation of angular momentum, find v_a in terms of v_p. (iii) Using conservation of energy, hence determine the value of v_a.

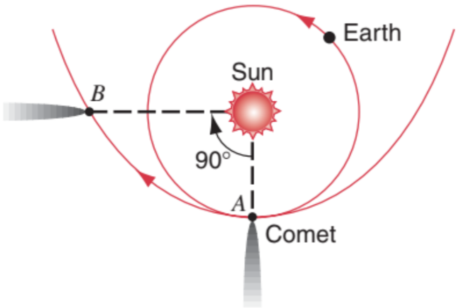
Problem 2.3. Find the magnitude of the gravitational force acting on a point mass m located at the center of a semi-circular frame (only the arc) with linear mass density μ and radius R.

Problem 2.4. A point mass m is placed along the axis of a ring M at a distance x from the centre of the ring, and released from rest. If the mass of the ring is distributed uniformly, and the radius of the ring is R, find the angular frequency of small oscillations of the point mass along the axis of the ring, assuming $x << R$.

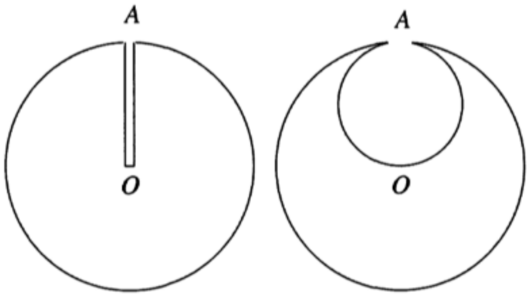
Problem 2.5 (SPhO 2019)**.** The internal and external radii of a uniform hollow sphere are r and R respectively. Taking the gravitational potential to be zero at infinity, what is the ratio of the gravitational potential at the point on the outer surface to that on the inner surface?

Problem 2.6 (Ricardo)**.** A spaceship is sent to investigate a planet of mass m_p and radius r_p. While hanging motionless in space at a distance $5r_p$ from the centre of the planet, the ship fires an instrument package with speed v_0. The package has mass m_i which is much smaller than the mass of the spaceship. The package is launched at an angle θ with respect to a radial line between the centre of the planet and the spaceship. For what angle θ will the package just graze the surface of the planet?

Problem 2.7 (HRK)**.** How long will it take a comet, moving in a parabolic path, to move from its point of closest approach to the Sun at A, through an angle of 90°, measured at the Sun, to B? Let the distance of closest approach to the Sun be equal to the radius of the Earth's orbit, assumed circular.



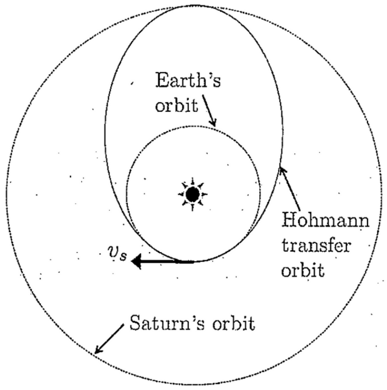
Problem 2.8 (Kevin Zhou)**.** A spaceship of titanium-devouring little green people has found a perfectly spherical homogeneous asteroid. A narrow trial shaft was bored from point A on its surface to the center O of the asteroid. At that point, one of the little green men fell off the surface of the asteroid into the trial shaft. He fell, without any braking, until he reached O, where he died on impact.



However, work continued, and the little green men started secret excavation of the titanium, in the course of which they formed a spherical cavity of diameter AO inside the asteroid. Then, a second accident occurred: another little green man similarly fell from point A to point O, and died. (i) Find the ratio of the impact speeds for the two men. (ii) Find the ratio of the times of impact for the two men.

Problem 2.9 (SPhO 2017)**.** A planet is in a circular orbit around a star of mass M. The star is much more massive than the planet. The star explodes symmetrically and instantaneously such that its outer envelope is pushed far beyond the orbit of the planet (and does not interfere with the planet). The remaining piece of the star is of mass M' (still much greater than the mass of the planet). Find the eccentricity of the new orbit of the planet.

Problem 2.10 (SPhO 2016)**.** A spacecraft is to be sent from Earth to Saturn. The Hohmann transfer is a manoeuvre that transfers the spacecraft from Earth's orbit around the Sun (assumed to be circular) to Saturn's orbit around the Sun (assumed to be circular also). This is done by pushing the spacecraft into an elliptical orbit that has its perihelion and aphelion touching the Earth's orbit and Saturn's orbit, respectively.



Through this problem, you may use the following symbols and constants in your working.

1. Earth's velocity around the Sun: $v_{\oplus} = 2.97 \times 10^4\,\text{m/s}$
2. Earth's orbital radius around the Sun: $r_{\oplus} = 1.50 \times 10^{11}\,\text{m}$
3. Saturn's orbital radius around the Sun: $r_{♄} = 1.43 \times 10^{12}\,\text{m}$
4. Mass of the Sun: $M_{\odot} = 1.99 \times 10^{30}\,\text{kg}$
5. Venus' orbital radius around the Sun: $r_{♀} = 1.08 \times 10^{11}\,\text{m}$
6. Venus' orbital period around the Sun: $T_{♀} = 0.615\,\text{years}$

(i) Assuming that the spacecraft is initially travelling at the velocity of the Earth around the Sun, calculate the change in velocity Δv for the spacecraft to transfer from Earth's orbit to the elliptical orbit. Ignore the Earth's and Saturn's gravitational forces. (ii) Calculate the time it takes for the spacecraft to travel from Earth's orbit to Saturn's orbit. (iii) The Cassini spacecraft was not sent to Saturn by a direct Hohmann transfer. Instead, it was first sent to Venus (via Hohmann transfer), where it gained speed twice, using gravity assists by Venus. The first gravity assist by Venus changed the spacecraft's velocity and transferred the spacecraft to a more eccentric orbit, such that the spacecraft returned to its orbital perihelion after Venus had completed its orbit twice, and the spacecraft received the second gravity assist by Venus at that point. Calculate the change in velocity of the spacecraft during the first Venus' gravity assist.

Problem 2.11 (Morin)**.** Let the Earth's radius be R, its average density be ρ, and its angular frequency of rotation be ω. Consider a long rope with uniform mass density extending radially from just above the surface of the Earth out to a radius of ηR. (i) Show that if the rope is to remain above the same point on the Equator at all times, then we must have

$$\eta^2 + \eta = \frac{8\pi G\rho}{3\omega^2}$$

(ii) Where does the tension along the rope achieve its maximum value?

Problem 2.12 (Ricardo)**.** The trajectory of a particle under the influence of a central force (not necessarily gravitational) is a circle of radius R. The position of the source of the central force is not known. Given that the maximum and minimum speeds of the particle are v_1 and v_2 respectively, determine the period T of the particle's orbit. (Hint: Kepler's 2nd Law holds for *all* central forces.)

Problem 2.13 (200 Puzzling Physics Problems)**.** A rocket is launched from and returns to a spherical planet of radius R so that its velocity vector on return is parallel to its velocity vector at launch. The angular separation at the center of the planet between the launch and arrival points is θ. How long does the flight take, if the period of a satellite flying around the planet just above its surface is T_0?

3 Advanced Problems

These problems are way too difficult to be tested in a modern-day SPhO. If you have completed all the previous problems and are down for a challenge, try these!

Problem 3.1 (Kevin Zhou)**.** We shall investigate the first, second and third cosmic velocities. Evaluate all your answers numerically, using the following values:

1. Mass of the Earth: $M_{Earth} = 5.97 \times 10^{24}\,\mathrm{kg}$
2. Mass of the Sun: $M_{Sun} = 1.99 \times 10^{30}\,\mathrm{kg}$
3. Radius of the Earth: $R_{Earth} = 6.37 \times 10^{6}\,\mathrm{m}$
4. Distance between the Sun and the Earth: $d_{Sun} = 1.50 \times 10^{11}\,\mathrm{m}$

(i) Starting from the Earth's surface, what is the minimum launch speed required to put a satellite into orbit around Earth? This is the first cosmic speed. Do not account for the rotation of the Earth. (ii) Now, accounting for the rotation of the Earth, what is this new minimum speed, and how should the satellite be launched? (iii) What is the minimum launch speed required for a rocket to escape the gravitational field of the Earth? This is the second cosmic speed. (iv) What is the minimum launch speed required for a rocket to leave the solar system? This is the third cosmic speed. How should the satellite be launched? Work in the approximation that $R_{Earth} << d_{Sun}$. (v) What is the minimum launch speed for a rocket to hit the Sun? Assume that you cannot make any adjustments to the rocket's path after launch. (vi) If subsequent adjustments are allowed, the minimum launch speed in (v) can be dramatically reduced. Find the new minimum launch speed if an infinitesimal adjustment later is allowed.

Remark. You should only be finding (iv)-(vi) difficult.

Problem 3.2 (Physics Cup)**.** A cannonball is launched from the Equator and subsequently hits the North Pole. Neglecting air resistance and the Earth's rotation, at what angle to the horizontal should the cannonball be fired to minimise the required speed? (The North Pole and the Equator differ by a latitude of 90°.)

Problem 3.3 (APhO 2017 T2)**.** An excellent gravitation problem. This problem is representative of the level of international Olympiads, so don't be too discouraged if you can't solve parts of it!

4 Appendix

4.1 Deriving the Total Energy of an Elliptical Orbit

Let us introduce the quantities below, which will be useful for the proof:

1. **Specific Angular Momentum:** $J = \frac{L}{m}$, defined as the angular momentum per unit mass.
2. **Specific Energy:** $\epsilon = \frac{E}{m}$, defined as the (total) energy per unit mass.

Clearly, since L and E are conserved, thus J and ϵ are conserved too.

Let the periapsis distance be r_1 and the apoapsis distance be r_2. By conserving J,

$$J = r_1 v_1 = r_2 v_2 \tag{38}$$

and by conserving 2ϵ at these two points,

$$2\epsilon = v_1^2 - \frac{2GM}{r_1} = v_2^2 - \frac{2GM}{r_2} \tag{39}$$

Thus, combining Equations (38) and (39),

$$\epsilon = \frac{J^2}{2r^2} - \frac{GM}{r} \quad \Rightarrow \quad \epsilon r^2 + GMr - \frac{J^2}{2} = 0 \tag{40}$$

whereby Equation (40) is satisfied by both $r = r_1$ and $r = r_2$.

By Vieta's formula,

$$\text{Sum of roots} = r_1 + r_2 = -\frac{GM}{\epsilon} = -\frac{GMm}{E} \tag{41}$$

Recall that $r_1 + r_2 = 2a$. Thus,

$$2a = -\frac{GMm}{E} \quad \Rightarrow \quad E = -\frac{GMm}{2a} \tag{42}$$

as desired. This concludes the proof.

4.2 A Discussion on Central Forces

In general, a **central force** is defined as any force that points **purely radially** towards or away from a fixed point in space. For instance, clearly, the gravitational force is a central force.

Since we care about the radial direction, let's turn our focus to polar coordinates. Writing N2L in polar coordinates,

$$F_r = m\left(\ddot{r} - r\dot{\theta}^2\right) \tag{43}$$

$$F_\theta = m\left(r\ddot{\theta} + 2\dot{r}\dot{\theta}\right) \tag{44}$$

Since we are dealing with a central force, $F_\theta = 0$ and $F_r = F(r)$. Equation (44) becomes

$$r\ddot{\theta} + 2\dot{r}\dot{\theta} = 0 \quad \Rightarrow \quad r^2\ddot{\theta} + 2r\dot{r}\dot{\theta} = 0 \quad \Rightarrow \quad \frac{d}{dt}\left(r^2\dot{\theta}\right) = 0 \quad \Rightarrow \quad r^2\dot{\theta} = \text{constant} \tag{45}$$

Recall that $L = mr^2\dot{\theta}$. Thus, this implies that L is conserved, as we expect.

With L, we may analyse the total energy of a system under the influence of a central force. The total energy is given by

$$E = K + U = \frac{1}{2}m\left(\dot{r}^2 + r^2\dot{\theta}^2\right) + U(r) = \frac{1}{2}m\dot{r}^2 + \left(U(r) + \frac{L^2}{2mr^2}\right) \tag{46}$$

Notice how we grouped the terms together. The term in the brackets can be defined as the **effective potential energy,** $U_{eff}(r)$:

$$U_{eff}(r) := U(r) + \frac{L^2}{2mr^2} \tag{47}$$

This is convenient because the effective potential energy only depends on r, while the other term, $\frac{1}{2}m\dot{r}^2$, only depends on $\dot{r}$. Hence, the term outside the brackets more closely resembles kinetic energy, while the term inside the brackets more closely resembles potential energy!

Remark. The $\frac{L^2}{2mr^2}$ term is actually kinetic energy, but it is more convenient to just treat it as part of the potential energy.

We can also note that $U_{eff}(r) = E - \frac{1}{2}m\dot{r}^2$. As $\frac{1}{2}m\dot{r}^2 \geq 0$, thus

$$U_{eff}(r) \leq E \tag{48}$$

hence, setting an upper bound for $U_{eff}(r)$.

This idea is useful for finding apsides. At the apsides, the radial distance is maximal/minimal, and hence $\dot{r} = 0$. Thus, $E = U_{eff}$.

Going back to forces, we can use the substitution $\dot{\theta} = \frac{L}{mr^2}$, thus Equation (43) becomes

$$F(r) = m\left(\ddot{r} - r\left(\frac{L}{mr^2}\right)^2\right) = m\left(\ddot{r} - \frac{L^2}{m^2r^3}\right) \tag{49}$$

which is a differential equation solely in r. We have reduced our problem in r and θ to just r!

However, this is still not easy to solve. (You may try it as an exercise if you aren't convinced how hard it is to solve this.) Thus, let's turn our focus to an ingenious substitution, called the **inverse radial position,** u:

$$u := \frac{1}{r} \tag{50}$$

Since $dr = -\frac{1}{u^2}\,du$, we may convert the position-derivative operator:

$$\frac{d}{dr} = -u^2\frac{d}{du} \tag{51}$$

We also know from COAM that

$$L = mr^2\frac{d\theta}{dt} \quad \Rightarrow \quad \frac{d}{dt} = \frac{L}{mr^2}\frac{d}{d\theta} = \frac{Lu^2}{m}\frac{d}{d\theta} \tag{52}$$

Remark. At this point, it is worth noting that Equations (51) and (52) alone make no mathematical sense. Instead, here, we are dealing with **operators** - the $\frac{d}{dr}$ and $\frac{d}{dt}$ are operators which will be applied to quantities.

Now, applying Equation (52), we differentiate r with respect to time:

$$\dot{r} = \frac{Lu^2}{m}\frac{d}{d\theta}\left(\frac{1}{u}\right) = -\frac{Lu^2}{m}\left(-\frac{1}{u^2}\frac{du}{d\theta}\right) = -\frac{L}{m}\frac{du}{d\theta} \tag{53}$$

Differentiating with respect to time once more,

$$\ddot{r} = \frac{Lu^2}{m}\frac{d}{d\theta}\left(-\frac{L}{m}\frac{du}{d\theta}\right) = -\frac{L^2u^2}{m^2}\frac{d^2u}{d\theta^2} \tag{54}$$

Thus, Equation (49) reduces to

$$F(u) = -\frac{L^2u^2}{m}\frac{d^2u}{d\theta^2} - \frac{L^2u^3}{m} \quad \Rightarrow \quad \frac{d^2u}{d\theta^2} + u = -\frac{m}{L^2u^2}F(u) \tag{55}$$

or, equivalently, using the idea of effective potential energy in Equation (47),

$$\frac{d^2u}{d\theta^2} = -\frac{m}{L^2}U'_{eff}(u) \tag{56}$$

This is a 2nd order ODE that we are more familiar with, and is commonly referred to as the **Binet Equation.**

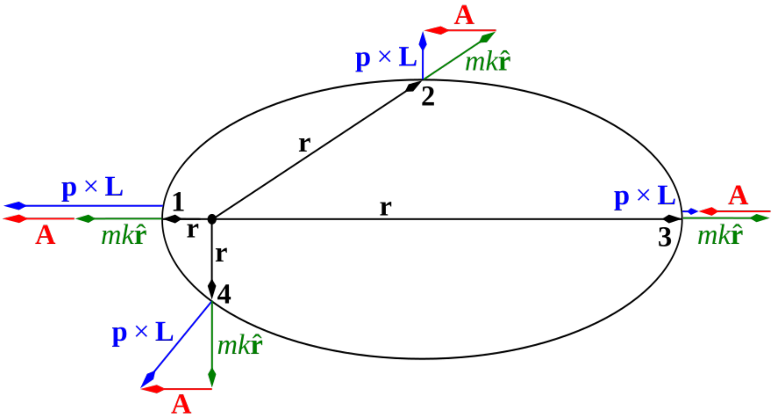
What remains is to substitute in $F(u)$ and solve for $u(\theta)$, and hence $r(\theta)$. You have solved for the polar equation of the trajectory of a particle under a central force!

4.3 Laplace-Runge-Lenz Vector

The **Laplace-Runge-Lenz vector**

$$\mathbf{A} = \mathbf{p} \times \mathbf{L} - GMm^2\hat{\mathbf{r}} \tag{57}$$

is a special quantity that is **conserved,** given a star of mass M at the origin, and a planet with radial position $\mathbf{r}$, momentum $\mathbf{p}$ and angular momentum $\mathbf{L}$.



To show that this is conserved, we consider the time-derivative of $\mathbf{A}$. Any conserved quantity will have its time-derivative equal to 0, so our goal is to prove that it is 0.

Recall that

$$\mathbf{p} = m\mathbf{v}, \ \ \mathbf{L} = m\mathbf{r} \times \mathbf{v}, \ \ \frac{d\mathbf{p}}{dt} = \mathbf{F}$$

Thus, we can evaluate

$$\mathbf{p} \times \mathbf{L} = m\mathbf{v} \times (m\mathbf{r} \times \mathbf{v}) = m^2\mathbf{v} \times (\mathbf{r} \times \mathbf{v}) = m^2\left(v^2\mathbf{r} - \mathbf{v}(\mathbf{r} \cdot \mathbf{v})\right) \tag{58}$$

where we have used the vector triple product for the last equality.

Now, we can differentiate

$$\frac{d\mathbf{A}}{dt} = \frac{d}{dt}\left(\mathbf{p} \times \mathbf{L} - GMm^2\hat{\mathbf{r}}\right) = \frac{d}{dt}\left(m^2\left(v^2\mathbf{r} - \mathbf{v}\left(\mathbf{r}\cdot\mathbf{v}\right)\right) - GMm^2\hat{\mathbf{r}}\right) \tag{59}$$

We shall simplify term by term:

$$\frac{d}{dt}\left(v^2\mathbf{r}\right) = \frac{d}{dt}\left(\left(\mathbf{v}\cdot\mathbf{v}\right)\mathbf{r}\right) = \left(2\mathbf{v}\cdot\mathbf{a}\right)\mathbf{r} + v^2\mathbf{v} \tag{60}$$

$$\frac{d}{dt}\left(\left(\mathbf{r}\cdot\mathbf{v}\right)\mathbf{v}\right) = \left(v^2 + \mathbf{r}\cdot\mathbf{a}\right)\mathbf{v} + \left(\mathbf{r}\cdot\mathbf{v}\right)\mathbf{a} \tag{61}$$

$$\frac{d\hat{\mathbf{r}}}{dt} = \frac{d}{dt}\left(\frac{\mathbf{r}}{r}\right) = \frac{\mathbf{v}}{r} - \frac{\mathbf{r}}{r^2}\frac{dr}{dt} = \frac{\mathbf{v}}{r} - \frac{\mathbf{r}}{r^2}\left(\hat{\mathbf{r}}\cdot\mathbf{v}\right) = \frac{\mathbf{v}}{r} - \frac{\left(\mathbf{r}\cdot\mathbf{v}\right)\mathbf{r}}{r^3} \tag{62}$$

Note that in Equation (62), the second equality is true from the definition of radial speed:

$$\frac{dr}{dt} = \frac{d}{dt}\left(\sqrt{\mathbf{r}\cdot\mathbf{r}}\right) = \frac{1}{2r}\frac{d\left(\mathbf{r}\cdot\mathbf{r}\right)}{dt} = \frac{\mathbf{r}\cdot\mathbf{v}}{r} = \hat{\mathbf{r}}\cdot\mathbf{v}$$

Now, recall that

$$\mathbf{a} = \ddot{\mathbf{r}} = \frac{\mathbf{F}}{m} = -\frac{GM}{r^2}\hat{\mathbf{r}} = -\frac{GM}{r^3}\mathbf{r} \tag{63}$$

Thus, the full simplification of Equation (59) is as such:

$$\begin{aligned}\frac{d\mathbf{A}}{dt} &= m^2\left(2\left(\mathbf{v}\cdot\mathbf{a}\right)\mathbf{r} - \left(\mathbf{r}\cdot\mathbf{a}\right)\mathbf{v} - \left(\mathbf{r}\cdot\mathbf{v}\right)\mathbf{a}\right) - GMm^2\frac{\mathbf{v}}{r} + GMm^2\frac{\mathbf{r}\cdot\mathbf{v}}{r^3}\mathbf{r} \\ &= m^2\left(-\frac{2GM}{r^3}\left(\mathbf{r}\cdot\mathbf{v}\right)\mathbf{r} + \frac{GM}{r}\mathbf{v} + \frac{GM}{r^3}\left(\mathbf{r}\cdot\mathbf{v}\right)\mathbf{r}\right) - GMm^2\frac{\mathbf{v}}{r} + GMm^2\frac{\mathbf{r}\cdot\mathbf{v}}{r^3}\mathbf{r} \\ &= m^2\left(-\frac{GM}{r^3}\left(\mathbf{r}\cdot\mathbf{v}\right)\mathbf{r} + \frac{GM}{r}\mathbf{v}\right) - GMm^2\frac{\mathbf{v}}{r} + GMm^2\frac{\mathbf{r}\cdot\mathbf{v}}{r^3}\mathbf{r} = 0\end{aligned}$$

which is exactly as desired, thus proving that $\mathbf{A}$ is a constant of motion.

This may seem extremely contrived, but the Laplace-Runge-Lenz vector can be elegantly used to solve some problems. For a very hard problem that uses this idea, you may look at Physics Cup 2021, Problem 2.